DEGREES OF FREEDOM AND THREE-CORNERED HATS

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Abstract

The three-cornered hat is a procedure for extracting the stabilities of three clocks when the only available information is the time or frequency differences between the clocks. To our knowledge, there has been no method of determining a confidence interval for such a stability estimate. In this paper, we present a method for determining the number of degrees of freedom of the estimate, which allows the assignment of a confidence interval to a three-cornered-hat stability estimate.

INTRODUCTION

The basic question that this paper will address is how to assign a confidence interval to a three-cornered-hat estimate for the stability of a clock. Assigning a confidence interval requires knowledge of the number of degrees of freedom of the estimate. While there exist empirical formulas for the (in general, fractional) number of degrees of freedom for an overlapping estimation of the Allan deviation, the authors know of no equivalent result for an estimate from a three-cornered hat. We present an analytic expression for the fraction of degrees of freedom that are lost through application of the three-cornered hat, which will allow the assignment of confidence intervals. In addition, we verify our analytic expression through the use of numerical simulations.

STABILITY METRICS

The timing community uses a variety of stability metrics for the characterization of frequency standards, clocks, and oscillators (these terms will be used interchangeably here). We will limit our discussion to the Allan deviation [1]. The Allan deviation is the average of the neighboring frequency differences that have been averaged for a given length of time τ. The formal definition is:

$$\sigma_y(\tau) = \frac{1}{2}\left\langle \left(\bar{y}_{t+\tau} - \bar{y}_t \right)^2 \right\rangle \qquad (1)$$

where $\bar{y}_t$ is the fractional frequency averaged over an interval τ, and the angled brackets indicate an average over all time. The factor of 1/2 is included to replicate the RMS for a white frequency data. For the remainder of the paper, we will suppress the subscript y in the Allan deviation and use the subscript to indicate a clock or time series label. This convention will be used for both the RMS and the Allan deviation.

It should be noted that if the Allan deviation is to be interpreted as the stability of a single clock or oscillator, a trusted (much more stable) reference is needed. In this case, one measures phase or frequency differences between the device under test and the trusted reference. In this situation, one can neglect the noise contribution from the reference and assign the noise in the measurements to the device under test.

DEGREES OF FREEDOM

When given a data set of phase or frequency measurements, one can estimate the stability of the oscillator by using Equation (1). An important next step is the evaluation of the confidence interval of the estimated Allan deviation. This question is addressed in a paper by Howe, Allan, and Barnes [2], where the authors arrive at empirical formulas for the number of degrees of freedom for an estimate of the Allan deviation. The empirical formulas require knowledge of the number of samples in the data set, the averaging interval, and the dominant noise type at that averaging interval.

Once the number of degrees of freedom are known, a confidence interval can be assigned by referring to the integrated Chi-Squared distribution for that number of degrees of freedom.

THE THREE-CORNERED HAT

It is often the case that one must measure a clock or oscillator without the luxury of a trusted, superior reference. In this case, a common technique is to use phase or frequency measurements between three (or more) oscillators in a procedure commonly referred to in the timing community as a three-cornered hat (TCH).

The first step in the TCH procedure is to take a time series corresponding to the difference between each possible clock pair combination and form an estimate of the Allan deviation for each of the three time series σ_{ij} using Equation (1). If one assumes that there are no correlations between the time series of the individual clocks (the "bare" clock, measured against a mythical perfect reference), one can form an estimate of each clocks frequency stability as follows:

$$\sigma_i^2 = \frac{1}{2}\left(\sigma_{ij}^2 + \sigma_{ik}^2 - \sigma_{jk}^2\right).$$ (2)

The subscripts i, j, and k refer to the three clocks.

THE GRUBBS ESTIMATOR

It turns out that the idea of the TCH estimate predates its use in the timing and clock community. An earlier work is that of Grubbs [3]. In that paper, Grubbs investigated the combination of three or more measurements in an effort to characterize uncertainties in the measured quantities and the measurement systems. The work dealt with Gaussian noise and used the RMS as the measure of the measurement uncertainties.

DEGREES OF FREEDOM FOR THE GRUBBS ESTIMATE

The work of Grubbs included investigations into the variance of the measurement uncertainty estimates. The calculations of these variances made heavy use of the fact that higher moments of a Gaussian distribution are determined by only the mean and RMS of the distribution. The resulting variance of the variance estimate is:

$$\mathrm{var}\left(\sigma_i^2\right) = \frac{1}{n-1}\left(2\sigma_i^4 + \sigma_1^2\sigma_2^2 + \sigma_1^2\sigma_3^2 + \sigma_2^2\sigma_3^2\right),$$ (3)

where n is the number of samples in the data set. It is clear that the variance of any estimate is larger than the variance if that estimate could have been made by measuring against a perfect reference. In addition, the variance of the estimate depends on the estimated stabilities of the two other processes.

DEGREES OF FREEDOM IN A THREE-CORNERED HAT

The form of Equation (3) suggests that one can use it along with the higher moments of the Gaussian distribution for clock i to arrive at the ratio of the number of degrees of freedom between the TCH estimate and the "bare" estimate of clock i. This ratio, which we are suggesting for use in the TCH with the Allan deviation, is:

$$\Gamma_i = \frac{2\sigma_i^4}{\left(2\sigma_i^4 + \sigma_1^2\sigma_2^2 + \sigma_1^2\sigma_3^2 + \sigma_2^2\sigma_3^2\right)}, \quad i=1,2,3. \tag{4}$$

Gamma is the fraction of the number of degrees of freedom that are left after the TCH procedure for clock i when one estimates its stability in the presence of two other clocks. The procedure for calculating a confidence interval for a TCH clock estimate is then a fairly simple extension of the current methods for a single clock. The first step is to perform a TCH and then estimate the dominant noise type for that averaging interval [5]. With information about the noise type, the user must then use the results of Howe, Allan, and Barnes [2] to calculate the number of degrees of freedom for the situation if the estimate had been made with respect to a perfect reference. The resulting number of degrees of freedom are then multiplied by Gamma (Equation 4) for that clock estimate at that averaging time. The confidence interval can then be assigned using standard methods of comparison to the integrated Chi-Squared distribution for that number of remaining degrees of freedom.

SIMULATIONS

In an attempt to verify Equation 4, and to illustrate several common Allan deviation estimation situations, we performed several simulations using synthesized clock data.

DESCRIPTION

Each experiment consisted of 1,000 realizations of a TCH procedure on three synthesized clock data sets. Each synthesized clock time series was 1,025 data points in length and represented a pure power-law noise (white phase, flicker phase, white frequency, flicker frequency, or random-walk frequency modulation). The data sets were generated using the algorithms of Kasdin and Walker [4].

For each realization, the Allan deviation was estimated. The clock data sets were differenced and a TCH was performed. The results were recorded for all 1,000 realizations of each experiment. The number of degrees of freedom were estimated for each averaging interval for the "bare" clocks (original data sets), as well as the TCH estimates for each clock. Each experiment was repeated for all five common power-law noise types.

As expected, the results for the "bare" clocks replicated the results of Howe, Allan, and Barnes [2].

RESULTS FOR THREE CLOCKS OF SIMILAR STABILITY

A common application of the TCH involves three clocks with (supposedly) similar stabilities. The results of one set of simulations for this situation are shown in Figure 1. The upper portion of the figure shows the number of degrees of freedom for the stability estimates of both the "bare" clocks and the TCH estimates. The lower portion of the figure shows the ratio of the number of degrees of freedom. This ratio should be Gamma from Equation (4), with all of the stabilities being equal. In this case, Gamma is expected to be 2/5, which is shown as the heavy line in the lower portion of the figure.
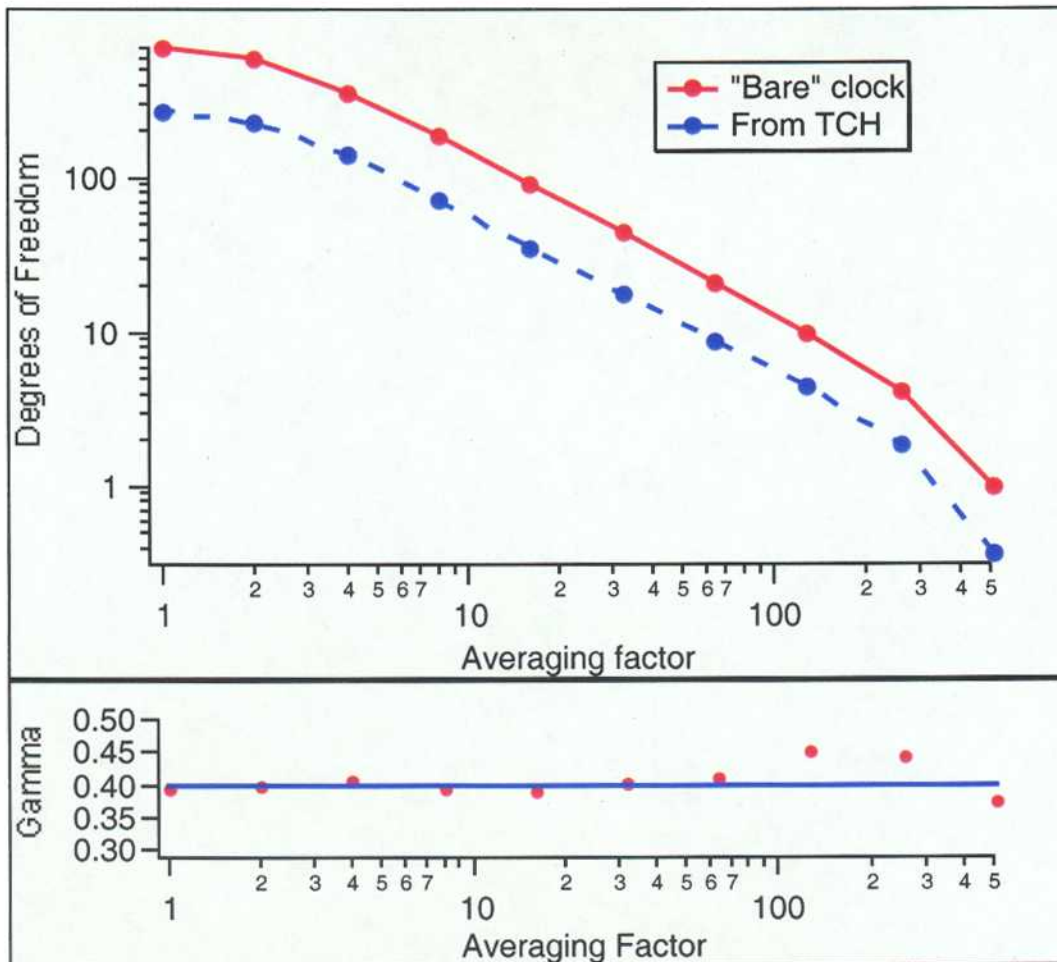
Figure 1: Degrees of freedom for a three-cornered hat (TCH) with clocks of equal stability. The red circles in the upper trace are the number of degrees of freedom for the underlying clocks used in the TCH. Blue circles show the number of degrees of freedom after application of the TCH. The lower trace is the ratio of the upper two traces, defined as Gamma in the text, along with a solid line that represents the prediction from Equation (4).

As noted previously, the results for the bare clock agree with the results of Howe, Allan, and Barnes [2]. The results for Gamma were unchanged for all common clock noise types (wp, fp, wf, ff, and rwf) and did not deviate from the predictions of Equation (4) until the number of remaining degrees of freedom were small. When the number of degrees of freedom in the TCH are small, there is an increasing probability that the procedure will fail, giving a negative variance estimate for the clock.

RESULTS FOR CLOCKS OF DIFFERING STABILITY

We also simulated the situation where one clock was being estimated with the aid of two other clocks that were half as stable. As in the previous section, the results for the "bare" clocks agreed with the results of Howe, Allan, and Barnes. The number of remaining degrees of freedom also agreed with Equation (4) until the number of remaining degrees of freedom were small. The results for the value of Gamma did not change with noise type.

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Figure 2 shows the results from a simulation with Clock 1 having being estimated with the aid of two other clocks with twice the level of instability. As predicted by Equation (4), the number of degrees of freedom are greatly reduced as compared with the simulations in the previous section.
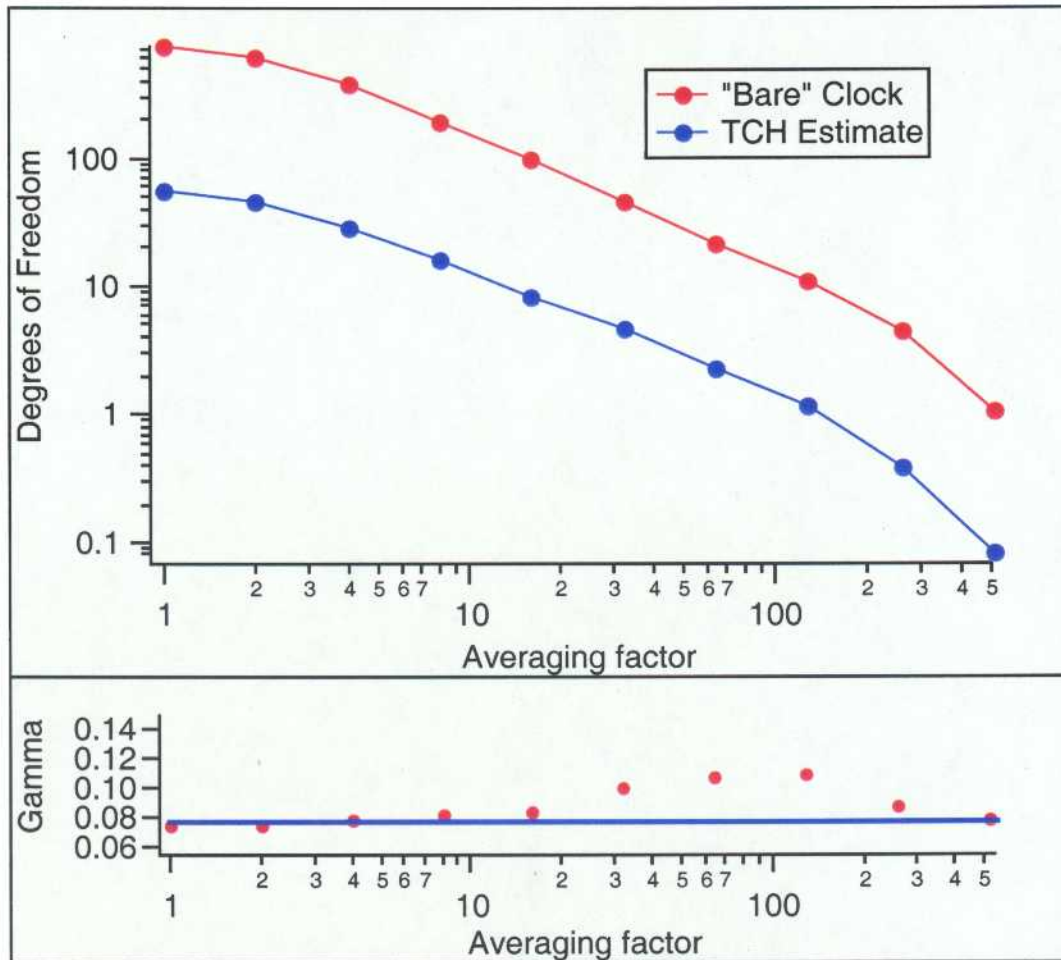

Figure 2: Degrees of freedom for a three-cornered hat (TCH) with clocks of differing stability. The results are for a clock whose stability is estimated with two clocks that are half as stable. The red circles in the upper trace are the number of degrees of freedom for the underlying clock used in the TCH. Blue circles show the number of degrees of freedom after application of the TCH. The lower trace is the ratio of the upper two traces, defined as Gamma in the text, along with a solid line that represents the prediction from Equation (4).

RANGES OF VALIDITY FOR THE THREE-CORNERED HAT

In both of the previous sections, we have seen that the simulations do not agree with the predictions for Gamma at large averaging factors. By looking at the fraction of time that the TCH fails as a function of averaging time and number of degrees of freedom, we have seen that one should demand a situation with 10 remaining degrees of freedom to expect a non-negative variance estimate. When the TCH estimate has 10 or more remaining degrees of freedom, the chance that the procedure will fail is well below 5%. It

is exactly the estimates where the number of remaining degrees of freedom falls below 10 that the predictions of Equation (4) are degraded in Figures 1 and 2.

AN INTERESTING ANALYTIC CASE

It is interesting to look at one additional case that arises in clock measurements. The situation is when one wishes to estimate the stability of a clock with two others that are less stable. The factor gamma for this situation can be found for the case when the two other clocks are a factor of N less stable ($\sigma_2 = \sigma_3 = N\sigma_1$). Gamma follows from Equation (4), and is

$$\Gamma = \frac{2}{2 + 2N^2 + N^4}.$$
(5)

Equation (5) highlights the need for designing experiments with long data collection times when one is estimating an oscillator's stability with lower quality references.

CONCLUSION

We have supplied an expression for the loss in the number of degrees of freedom suffered by performing a three-cornered-hat estimation of a clock's Allan deviation. This expression allows the assignment of confidence intervals for the resulting stability estimates. We have also illustrated the applicability and ranges of validity for this expression and the three-cornered-hat estimates themselves.

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